Honors AP Calculus / Mr. Hansen
11/10/2008

Name: KEY

Bonus (for Mr. Hansen's use only): add 5 for all students (late start)

Test through Chapter 5 (Calculator Required)

Rules

- You may not write calculator notation anywhere unless you cross it out. For example, fnInt(X^2,X,1,2) is not allowed; write $\int_1^2 x^2 dx$ instead.
- Adequate justification is required for all free-response questions.
- Unless a blank is provided for you to write your answer, all final answers should be *circled or boxed.*
- Decimal approximations must be correct to at least 3 places after the decimal point.

Part I Each correct answer in Part I is worth 3 points. Unlike the scoring on the AP exam, there is no penalty today for wrong guesses (other than the loss of the 3 points, of course.)

In the small blank provided, write a capital letter A if the statement is always true, S if it is sometimes true, or N if it is never true. There is no partial credit.

A 1. A function that is differentiable on [a, b], where $b > a$, is Riemann integrable on [a, b].

S 2. The trapezoid rule estimate (using infinite-precision arithmetic) of the definite integral of a cubic polynomial function equals the true exact value of the integral.

N 3. For a continuous, odd function f, $\int_{-a}^{a} f(x)dx > 0$.

A 4. The Simpson's Rule estimate (using infinite-precision arithmetic) of the definite integral of a quadratic function equals the true exact value of the integral.

A 5. For a constant function y, $dy = \Delta y$.

N 6. For a differentiable function $y = f(x)$ with upward concavity, such as an exponential function with a base greater than 1, $dy > \Delta y$ even if $\Delta x < 0$.

S 7. A differentiable, strictly monotone function on $\Re$ has a point of inflection.

A 8. The trapezoid rule gives the same result that one would obtain by averaging the left and right endpoint rule estimates defined on the same partition.

A 9. For a standard midpoint rule Riemann sum involving independent variable x, the norm of the partition equals Δx.

A 10. If f is a continuous function of t and an initial condition (t_0, y_0) has been specified, then there exists a unique solution to the differential equation $\frac{dy}{dt} = f(t)$.

Part II (16 points)

11. On the reverse side, state both versions of the Fundamental Theorem of Calculus, and prove that each implies the other. In the chains of equalities, provide a short reason (a few words will suffice) for each step in the chain. *Note:* These reasons were not provided in the Internet versions of the proofs, but we discussed them briefly in class.

For both, let f be cont. on $[a,b]$.

FTC 1: $\int_a^b f(x)\,dx = G(b) - G(a)$ for any antiderivative G ; i.e., where $G' = f$

FTC 2: $\frac{d}{dx} \int_k^x f(t)\,dt = f(x)$ for k a constant $\in [a,b]$

FTC 1 $\Rightarrow$ FTC 2 :

$$
\begin{aligned}
\frac{d}{dx} \int_k^x f(t)\,dt &= \frac{d}{dx}\left[G(x) - G(k)\right] && \text{by subst.; FTC1 (where } G' = f) \\
&= G'(x) - \frac{d}{dx}\left[G(k)\right] && \text{deriv. of diff. = diff. of derivs.} \\
&= G'(x) - 0 && \text{deriv. of a const.} \\
&= G'(x) = f(x) \quad \blacksquare && \text{by def. of } G
\end{aligned}
$$

FTC2 $\Rightarrow$ FTC 1 :

Let $H(x) = \int_a^x f(t)\,dt$, an accumulator fcn. By FTC2, $H' = f$, so that H is an antideriv. of f. Since any antideriv. G (of f) can be written as $G(x) = H(x) + C$, we have $H(x) = G(x) - C$ for arbitrary constant C.

$$
\begin{aligned}
\int_a^b f(x)\,dx &= \int_a^k f(x)\,dx + \int_k^b f(x)\,dx && \text{by rules of def. integrals} \\
&= -\int_k^a f(x)\,dx + \int_k^b f(x)\,dx && \text{" " " " "} \\
&= \int_k^b f(x)\,dx - \int_k^a f(x)\,dx && \text{by commutativity} \\
&= H(b) - H(a) && \text{by def. of accum. fcn. } H \\
&= [G(b) - C] - [G(a) - C] && \text{by subst.} \\
&= G(b) - G(a) \quad \blacksquare && \text{by alg.}
\end{aligned}
$$

Part III (8 points for each numbered problem)

12. An automobile is moving along a straight line with velocity (meters per second) given by $v(t) = 2\cot^{-1}(\ln|\sin t|)$ for $3.5 < t < 6$.

(a) Is v continuous on the given interval? yes

(b) Let $s(t)$ denote the automobile's position at time t. Write an equation that gives the automobile's position (in meters) at time $t = 5$ seconds, if it is known that $s(4) = 2.882$.

$$s(5) = 2.882 + \int_4^5 2\cot^{-1}(\ln|\sin t|)\,dt$$

13. Under the same conditions as #12b, find $s(5)$ correct to 3 decimal places. Answer: $-.126$ m

14. Calculate the exact value (no decimals) of $\int_{-2}^{3}(x^2+3x-2)dx$, showing your work. You may, of course, use your calculator to check your answer at the end.

$$\int_{-2}^{3}(x^2+3x-2)\,dx = \left(\frac{x^3}{3} + 3\frac{x^2}{2} - 2x\right)\Big|_{-2}^{3}$$
$$= \frac{3^3}{3} + 3\left(\frac{9}{2}\right) - 6 - \left(\frac{-8}{3} + 3\left(\frac{4}{2}\right) - 2(-2)\right)$$
$$= 9 + \frac{27}{2} - 6 + \frac{8}{3} - 6 - 4$$
$$= \frac{55}{6}$$

15. For the function given in #14, use Simpson's rule with 7 mesh points (i.e., 6 intervals) to estimate the value of the integral on the interval [–2, 3]. Show your work.

Let $y = f(x) = x^2+3x-2$.

$$\text{Simpson's Rule est.} = \frac{1}{3}\Delta x\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6\right)$$
$$= \frac{1}{3}\left(\frac{5}{6}\right)\Big(f(-2) + 4f\left(-2+\frac{5}{6}\right) + 2f\left(-2+\frac{10}{6}\right) + 4f\left(-2+\frac{15}{6}\right) + 2f\left(-2+\frac{20}{6}\right) + 4f\left(-2+\frac{25}{6}\right) + f\left(-2+\frac{30}{6}\right)\Big)$$
$$= \frac{5}{18}\big(-4 + 4(-4.139\ldots) + 2(-2.888\ldots) + 4(-.25) + 2(3.\overline{77}) + 4(9.194\ldots) + 16\big)$$
$$= \frac{5}{18}\left(-4 - 16.\overline{5} - 5.\overline{7} - 1 + 7.\overline{5} + 36.\overline{7} + 16\right)$$
$$= \frac{5}{18}(33)$$
$$= \frac{55}{6} \text{ or approx. } 9.167$$

16-18. Let $Q(x) = e^x - x\ln x + x - x^3/3$, defined only on the domain [.5, 2]. Let $f(x) = dQ/dx$.

16.(a) Find dQ.

$$dQ = \left(e^x - \left[\cancel{\tfrac{x}{x}} + \ln x\right] + \cancel{1} - \tfrac{1}{3}\cdot 3x^2\right) dx$$

$$dQ = \left(e^x - \ln x - x^2\right) dx$$

(b) Use your calculator to approximate $\int_{.5}^{2} (e^x - \ln x - x^2)dx$. Answer: 2.882

17.(a) Find an exact algebraic expression (no integral signs, no decimals) for $\int_{.5}^{2} (e^x - \ln x - x^2)dx$. *Hint:* Make use of part #16(a).

$$\int_{.5}^{2} f(x)\,dx = Q(2) - Q(.5) \quad \text{by FTC1}$$

$$= e^2 - 2\ln 2 + 2 - \frac{8}{3} - \left(e^{.5} - .5\ln .5 + .5 - \frac{.5^3}{3}\right)$$

(b) Fill in the blanks: The Mean Value Theorem, applied to function Q, guarantees that

$\exists c \in$ (0.5, 2) such that $\dfrac{Q(b)-Q(a)}{b-a} =$ $Q'(c) = f(c)$.

18. Explain precisely how the equation in (~~a~~ b) implies that there exists a point c in the domain of Q such that the area under f from .5 to 2 equals 1.5 f(c).

Note: $c \in (a,b) = (.5, 2) \Rightarrow c \in [.5, 2] = \mathcal{D}_Q$. ↑ ital.

$$\frac{Q(b) - Q(a)}{b - a} = Q'(c) = f(c) \quad \text{where } a = .5,\ b = 2$$

$$\frac{Q(b) - Q(a)}{1.5} = f(c)$$

$$\underbrace{Q(b) - Q(a)} = 1.5\, f(c) \quad \square$$

This is the area under f from .5 to 2 by FTC 1,

since $\int_{.5}^{2} f(x)\,dx = Q(2) - Q(.5)$.

We established in #16(a) that Q is an antiderivative of f.