Mr. Hansen
§7-5 #8 Solution
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(a) $$\frac{1}{y(10-y)} = \frac{A}{y} + \frac{B}{10-y}$$

Mult. through by $y(10-y)$ to get

$$1 = A(10-y) + By$$

If this is to be an identity for y, the equation should hold even when $y=0$ or $y=10$. Ignore, for now, the fact that either value is excluded by the original fractional equation.

If $y=0$:

$$1 = A(10-0) + B(0)$$
$$1 = 10A$$
$$A = \frac{1}{10}$$

If $y=10$:

$$1 = A(0) + B(10)$$
$$1 = 10B$$
$$B = \frac{1}{10}$$

Test our solution:

$$\frac{1}{y(10-y)} \stackrel{?}{=} \frac{\frac{1}{10}}{y} + \frac{\frac{1}{10}}{10-y}$$

$$\frac{1}{y(10-y)} \stackrel{?}{=} \frac{1}{10}\left(\frac{1}{10-y} + \frac{1}{y}\right)$$

$$= \frac{1}{10}\left(\frac{y}{y(10-y)} + \frac{10-y}{y(10-y)}\right)$$

$$= \frac{1}{10}\left(\frac{y+10-y}{y(10-y)}\right)$$

$$= \frac{1}{10}\left(\frac{10}{y(10-y)}\right)$$

$$= \frac{1}{y(10-y)} \text{ as claimed } \checkmark$$

[The solution works, despite the quibble noted above.]

(b) $\int \frac{1}{y(10-y)}\,dy = 3\int dx$

$$\int \left(\frac{\frac{1}{10}}{y} + \frac{\frac{1}{10}}{10-y} \right) dy = 3x + C$$

$$\frac{1}{10}\left(\int \frac{1}{y}\,dy + \int \frac{1}{10-y}\,dy \right) = 3x + C$$

$$\frac{1}{10}\left(\ln|y| - \ln|10-y| \right) = 3x + C$$

$$\ln|y| - \ln|10-y| = 30x + C_1 \quad \text{where } C_1 = 10C$$

[Since both y and $10-y$ are usually positive (by analogy with #7 and the general situation for logistic growth), the absolute value bars can usually be removed. However, <u>never</u> simply discard the absolute value bars without having a good reason.]

$$\ln\left| \frac{y}{10-y} \right| = 30x + C_1$$

$$\left| \frac{y}{10-y} \right| = e^{30x + C_1}$$

$$\left| \frac{y}{10-y} \right| = e^{30x} e^{C_1} = C_2 e^{30x} \quad \text{where } C_2 = e^{C_1}$$

$$y = (10-y) C_3 e^{30x} \quad \text{where } C_3 = \pm C_2$$

$$y = 10C_3 e^{30x} - yC_3 e^{30x}$$

$$y(1 + C_3 e^{30x}) = 10 C_3 e^{30x}$$

$$y = \frac{10C_3 e^{30x}}{1 + C_3 e^{30x}}$$ [now divide through by $C_3 e^{30x}$]

$$y = \frac{10}{\frac{1}{C_3 e^{30x}} + 1}$$

$$y = \frac{10}{1 + \frac{1}{C_3} e^{-30x}}$$

$$y = \frac{10}{1 + ke^{-30x}} \quad \text{where } k = \frac{1}{C_3} = \pm\frac{1}{C_2} = \pm\frac{1}{e^{C_1}}$$

$$= \pm e^{-C_1} = \pm e^{10C}$$

(c) Clearly, since there was nothing special about the "3" and "10" in #8, the diffeq. from #7 can be solved in a similar fashion. In #7, we had diffeq.

$$\frac{dP}{dt} = P(a - bP) \quad \text{for } a \approx .028026, \; b \approx .000079275.$$

Rewrite this as

$$\frac{dP}{dt} = bP\left(\tfrac{1}{b}a - \tfrac{1}{b}(bP)\right)$$

$$\frac{dP}{dt} = bP\left(\tfrac{a}{b} - P\right),$$ which is now in the same form as #8, except that $\frac{a}{b}$ plays the role of "10," and b plays the role of "3."

General solution:

$$P = \frac{\frac{a}{b}}{1 + ke^{-at}}$$

Since $P = 131.7$ when $t = 0$,

$$131.7 = \frac{\frac{a}{b}}{1+k} \Rightarrow 1 + k = \frac{\frac{a}{b}}{131.7}$$

$$\Rightarrow k = \frac{\frac{a}{b}}{131.7} - 1 \approx 1.68435$$

(d) Let $\hat{P}(t) = \dfrac{\frac{a}{b}}{1 + ke^{-at}}$

where a, b, k are constants as defined above.

By calc.,

$\hat{P}(10)$ =	(pop. est. for 1950) ≈	155.556...	million
$\hat{P}(20)$ =	(" " " 1960) ≈	180.223...	"
$\hat{P}(30)$ =	(" " " 1970) ≈	204.756...	"
$\hat{P}(40)$ =	(" " " 1980) ≈	228.2307...	"
$\hat{P}(50)$ =	(" " " 1990) ≈	249.876...	"

(d) (cont'd.)

Euler's Method (using a step size of $\Delta t = 1$ yr.) gives the results shown in the first "Euler" column on p. 718, with the exception that the entry for 1990 should be 249.9... (full value: 249.99771612514) instead of 249.4..., an apparent typo.

It is not surprising that the Euler results track the logistic model results so closely, since the logistic model from 1940 to 1990 is passing through a rather linear portion of the graph. Because of strong immigration, however, U.S. population growth since about 1990 has been <u>much</u> greater than either the Euler model or the logistic model would predict. Both models predict population passing the 300 million mark in approximately 2020, but in fact, that milestone was reached in 2006.

This exercise illustrates the danger in <u>extrapolating</u> from a model in order to predict the future. The mere fact that two models agree closely gives no assurance that they will conform well with reality.

Link for more reading: www.300millionamericans.org/BlaineHarden_101206.pdf

That's a lower case L.