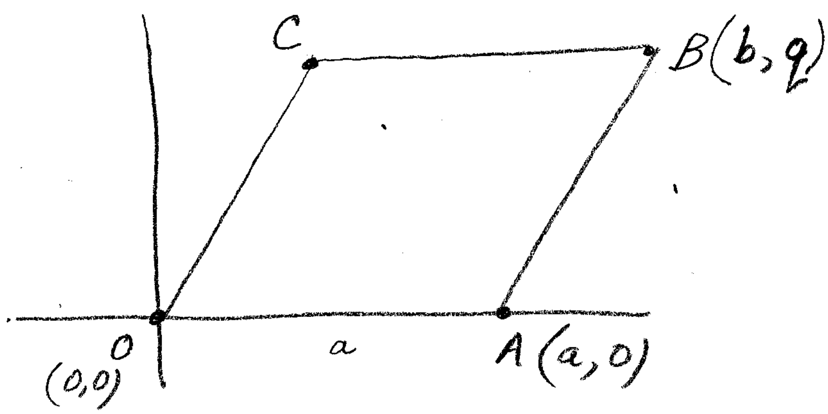
Mr. Hansen
Geom. 1/29/2009

Solution to Sample Quiz I

Given: Rhombus OABC
Prove: $\overline{OB}$ and $\overline{AC}$ are $\perp$ but not necessarily $\cong$



Wlog, place one vertex of rhombus at $(0,0)$ and let each side have length a, where $a > 0$. Wlog, let $b \geq a$. (Reorient the rhombus if necessary so that B lies on or to the right of the vertical line $x = a$ that passes through A.)

Since AB must equal a [def. rhombus], we can solve for q. By Pythag. Thm.,

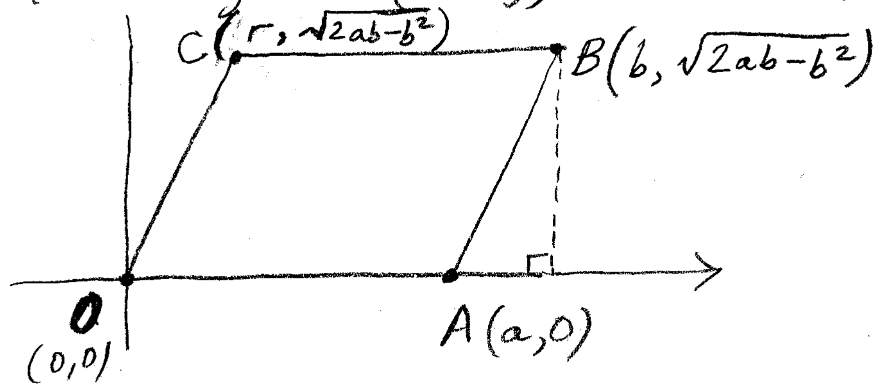
$AB = a = \sqrt{(b-a)^2 + q^2} = \sqrt{b^2 - 2ab + a^2 + q^2}$.

Square both sides to get $a^2 = b^2 - 2ab + a^2 + q^2$

$$0 = b^2 - 2ab + q^2$$

$$2ab - b^2 = q^2$$

Therefore, $q = \sqrt{2ab - b^2}$, and we can redraw the diagram (wlog) as follows:



The y-coordinate of C is $\sqrt{2ab-b^2}$ by inspection, since we know $\overline{BC} \parallel \overline{OA}$.

The x-coordinate of C, shown as r above, must be such that $BC = a$. Since the y-coordinates of C and B are the same, $BC =$ (difference of x-coordinates only). Therefore,

$$\begin{aligned} BC &= b - r \\ a &= b - r \\ a - b &= -r \\ b - a &= r \end{aligned}$$

Now, we know the coordinates of $O(0,0)$, $A(a,0)$, $B(b, \sqrt{2ab-b^2})$, and $C(b-a, \sqrt{2ab-b^2})$. [On the real quiz, you might be given this or something similar as a starting point.] The rest of the proof is easier.

Slope of $\overline{OB} = \dfrac{\Delta y}{\Delta x} = \dfrac{\sqrt{2ab-b^2}}{b}$

Slope of $\overline{AC} = \dfrac{\Delta y}{\Delta x} = \dfrac{-\sqrt{2ab-b^2}}{a-(b-a)} = \dfrac{-\sqrt{2ab-b^2}}{2a-b}$

Multiply slopes to get $\dfrac{-\sqrt{2ab-b^2}\sqrt{2ab-b^2}}{b(2a-b)} = \dfrac{-(2ab-b^2)}{2ab-b^2} = -1.$

Since the product is -1, the slopes are opp. reciprocals, which implies $\overline{OB} \perp \overline{AC}$. ▯

What about the diagonal lengths?

Use distance formula:

$$
\begin{aligned}
OB &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\
&= \sqrt{b^2 + \left(\sqrt{2ab-b^2}\right)^2} \\
&= \sqrt{b^2 + (2ab-b^2)} \\
&= \sqrt{2ab}
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
AC &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\
&= \sqrt{[a-(b-a)]^2 + \left(-\sqrt{2ab-b^2}\right)^2} \\
&= \sqrt{(2a-b)^2 + (2ab-b^2)} \\
&= \sqrt{4a^2 - 4ab + b^2 + 2ab - b^2} \\
&= \sqrt{4a^2 - 2ab} \\
&= \sqrt{2a(2a-b)}
\end{aligned}
$$

It does not appear that $OB = AC$. It could happen, but setting $OB = AC$ would be true iff

$$
\begin{aligned}
2ab &= 2a(2a-b) \\
b &= 2a-b \\
2b &= 2a \\
b &= a
\end{aligned}
$$

In other words, $OB = AC$ iff $b = a$, which would make the vertices $O, A, B,$ and C work out to be $(0,0)$, $(0,a)$, (a,a), and $(0,a)$ — in other words, the vertices of a square. Not all rhombuses are squares, which is how we know that $\overline{OB}$ need not be congruent to $\overline{AC}$. ▯