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Geom. Class Notes
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Just in time for Halloween!
It's the world's most deadly theorem!

But first, a lemma...

Lemma: Odd integers have odd squares, and even integers have even squares.

Paragraph proof:

<u>Case I (odd)</u>

- Wlog, let m be an odd integer. Then m is of the form $2k+1$, where k is some integer. In other words, m is 1 larger than the obviously even integer $2k$.
- Therefore,
$$\begin{aligned} m^2 &= (2k+1)^2 \\ &= (2k)^2 + 4k + 1 \\ &= 4k^2 + 4k + 1 \\ &= 4(k^2+k) + 1, \end{aligned}$$
which is 1 more than the obviously even integer $4(k^2+k)$. Thus m^2 is odd. $\square$

<u>Case II (even)</u>

- Wlog, let n be an even integer. Then n is of the form $2k$, where k is some integer, since 2 is a factor of any even integer.
- Therefore, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is 2 times the integer $2k^2$. Thus n^2 is even. $\square$

And now, the deadly theorem itself!

Theorem: $\sqrt{2}$ is irrational

Proof: Assume (bwoc) that $\sqrt{2}$ is rational. Then, by def. of rational, $\sqrt{2}$ can be written as a fraction $\frac{p}{q}$, where p and q are integers. Moreover, wlog, we may assume that p and q are not both even, for we can remove as many common factors of 2 as necessary from each. (For example, if the numerator were $2^{17}i$ and the denominator were $2^{17}j$, we would merely divide through by 2^{17} and take $\frac{p}{q}$ as the simplified fraction $\frac{i}{j}$.)

- Summary: $\sqrt{2} = \frac{p}{q}$ for integers p, q, where p, q are not both even.
- By algebra, $2 = \frac{p^2}{q^2}$

  $2q^2 = p^2 \Rightarrow p^2$ is even

  (since 2 times any integer is even).
- By the lemma on page 1, p^2 even $\Rightarrow p$ even $\Rightarrow$ p can be written as $2k$ for some integer k.
- By substitution and more algebra,

$$2q^2 = p^2$$
$$2q^2 = (2k)^2$$
$$2q^2 = 4k^2$$
$$q^2 = 2k^2 \Rightarrow q^2 \text{ is even}$$

- By the lemma again, q^2 even $\Rightarrow q$ even. Thus p and q are both even. $(\rightarrow\leftarrow)$ $\blacksquare$