Honors AP Calculus / Mr. Hansen
4/15/2010

Name: KEY

Test through §12-6 (No Calculator Permitted)

The time limit for the first question, which is an AP-style free-response question, is 15 minutes. You may not work on any other problems during this time. After 15 minutes, you will be given a set of 7 AP-style multiple-choice questions, for which the time limit is also 15 minutes. In the unlikely event that you finish those questions early, you may return to the free-response question if you wish.

After 30 minutes, all AP-style questions will be collected and replaced with non-AP questions. You should spend the remaining 20 minutes of the period working on the non-AP questions.

Scoring: 30 points (curved) for Part I, 30 points (curved) for Part II, 40 points (not curved) for Part III. There is a penalty for wrong guesses only in the middle section (AP multiple-choice).

Part I: Free Response in AP Style (30% of test score).

1. (a) Use the fact that $\dfrac{1}{x} = \dfrac{1}{1-(1-x)}$ to write x^{-1} as a geometric series.

$$\frac{1}{1-(1-x)} = 1 + (1-x) + (1-x)^2 + \cdots$$
$$= \sum_{n=0}^{\infty} (1-x)^n$$

- $\sum (1-x)^n$ or at least 2 valid terms, at least one of which includes $(1-x)$
- All terms valid, "…" or $\sum_{n=0}^{\infty}$ shown clearly

(b) State the common ratio, radius of convergence, and interval of convergence for the series in part (a).

- Common ratio $= 1-x$
- Radius of convergence $= 1$
- Interval of convergence $= (0, 2)$ [or $0 < x < 2$]

all exactly as shown

(c) Integrate your series in (a) term-by-term to develop a series expansion for ln x that is almost entirely in powers of $(1-x)$.

- Expression equiv. to: $\ln x = C + x + \frac{(1-x)^2}{-2} + \frac{(1-x)^3}{-3} + \ldots$

☆Note: 0/2 if omitting "..."

- Concludes that $C = -1$ (work optional) and gives answer similar to

$$-1 + x - \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - \ldots$$

or $$-(1-x) - \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - \ldots$$

☆ Note: 0/2 if omitting C; 1/2 if giving correct second answer without showing awareness of C

(d) For the series in part (c), find the interval of convergence, paying special attention to the behavior at the endpoints of the interval.

It is a fact that the interval of convergence will include a superset of $(0,2)$, the interval stated in part (b). This should be stated, or the interval should be found by ratio method:

$$\lim_{n\to\infty}\left|\frac{t_{n+1}}{t_n}\right| = \lim_{n\to\infty}\left|\frac{(1-x)^{n+1}}{-(n+1)} \cdot \frac{-n}{(1-x)^n}\right| = |1-x| \lim \frac{n}{n+1}$$

$$= |1-x| \overset{\text{want}}{<} 1$$

$\Rightarrow x \in (0,2)$, open int. of convergence

<u>However</u>, for this rubric, there are only 2 pts. checked, as follows:

- When $x=0$, series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots$, <u>divergent</u> harmonic series (or other suitable reason)
- When $x=2$, series is $-1 + 2 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$, <u>convergent</u> alternating harmonic series (or other suitable reason)

☆ Note: 1/2 if stating $(0,2]$ without giving reasons.

Part II: Multiple Choice in AP Style (30% of test score).

2. [See AP Calculus Course Description at

   http://apcentral.collegeboard.com/apc/public/repository/ap-calculus-course-description.pdf

   for this question. It is question #4 on page 28, and the answer is D.]

3. [See AP Calculus Course Description at

   http://apcentral.collegeboard.com/apc/public/repository/ap-calculus-course-description.pdf

   for this question. It is question #9 on page 30, and the answer is E.]

4. If the series $\sum_{i=1}^{\infty} t_i$ converges absolutely, then . . .

   (A) the series may or may not converge
   → (B) the series surely converges, but the error after n terms may or may not be bounded by $|t_{n+1}|$
   (C) the series surely converges, and the error after n terms is surely bounded by $|t_{n+1}|$
   (D) the series $\sum_{i=1}^{\infty} (-1)^{i+1} |t_i|$ may diverge
   (E) the series $\sum_{i=1}^{\infty} (-1)^{i+1} |t_i|$ surely diverges

[Explanations for #4–8 are the same as on the 4/14 scoring rubric.]

Conversion chart:

Free response (Part I)

Let x = raw score (0 to 9).
Scaled score (out of 30) equals $\frac{36+8x}{3}$.

Multiple choice (Part II)

Let $y = 4(\#\text{right}) - 1(\#\text{wrong}) + 0(\#\text{omitted})$.
Note that y can be anywhere from -7 to $+28$.
Scaled score (out of 30) equals $\frac{96+7x}{8}$.

5. Let n be a fixed positive integer. "Expression 1" refers to $\sum_{k=1}^{n} \frac{1}{k}$, and "Expression 2" refers to $\sum_{k=n+1}^{\infty} \frac{1}{k}$.

Which expression is greater?

(A) Expression 1, since n can be arbitrarily large.
(B) Expression 2, since Expression 1 is finite but Expression 2 is infinite.
(C) Neither, since Expression 1 is finite and Expression 2 converges to a finite value.
(D) Answering the question requires knowing the value of n.
(E) The answer to the question cannot be determined, even if the value of n is known.

6. The series $\sum_{n=1}^{\infty} (-1) \frac{1}{n + 5{,}000{,}000}$ converges to

(A) 0
(B) a negative value between -1 and 0
(C) a positive value between 0 and 1
(D) a positive value greater than or equal to 1
(E) DNE (not convergent)

7. Where does the power series $\sum_{n=0}^{\infty} (-1)^n \frac{n^{99} x^n}{n!}$ converge?

(A) $(-1, 1)$
(B) $[-1, 1)$
(C) $(-\frac{1}{e}, \frac{1}{e})$
(D) $[-\frac{1}{e}, \frac{1}{e})$
(E) $(-\infty, \infty)$

8. The sine of 1 radian, if estimated with a third-degree Taylor polynomial centered on $x = 0$, will be accurate to within what error tolerance? Choose the smallest value that serves as an upper bound for the absolute value of the error. (For example, if the absolute value of the error is bounded by 0.008, you would choose C, since $0.008 < 0.01$, and you would avoid D and E, since they are too small. You would also avoid A and B, since although they are upper bounds, they are too large. C is the *smallest* value that serves as an upper bound for the size of the error.)

(A) 0.5
(B) 0.1
(C) 0.01
(D) 0.001
(E) 0.0001

Part III: Free Response, Non-AP (40% of test score).
Point values are shown in parentheses in the left margin.

9.(a) State the radii of convergence (not intervals of convergence) for the standard Maclaurin series for
(6) exp(x), sin x, and cos x.

(2 ea.) exp(x): ∞ sin x: ∞ cos x: ∞

(b) Explain why it is no surprise that these series are convergent not only on $\Re$, but on the entire complex
(3) plane.

"Radius of convergence" reminds us that convergence occurs not only along a real interval but on a disk in $\mathbb{C}$; infinite radius means convergent everywhere.

(c) "Formal substitution" means plugging in without
(4)

regard for the meaning of the symbols.

(d) Use formal substitution to write cos($2i$) as a series of constants. **Simplify your answer for full credit.**
(5) The symbol i, as you know, is the complex unit, the principal root of the quadratic equation $x^2 + 1 = 0$.

$$\cos 2i = 1 - \frac{(2i)^2}{2!} + \frac{(2i)^4}{4!} - \frac{(2i)^6}{6!} + \cdots$$

$$= 1 + \frac{4}{2!} + \frac{16}{4!} + \frac{64}{6!} + \cdots = \cosh 2$$

[In fact, wlog $\cos ix = \cosh x$.]

(e) Prove, formally, that for any value θ, whether real or complex, $e^{i\theta} = \cos\theta + i\sin\theta$. This result is called
(10) *Euler's formula.* If you need to continue on the reverse side, write "OVER" in large letters.

Using the 4/14 test as a lemma, $e^{i\theta} = \cosh i\theta + \sinh i\theta$.

By part (d), $\cosh i\theta = \cos[i(i\theta)] = \cos(-\theta) = \cos\theta$.

Using Maclaurin's sine series, $i \sin\theta = i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$

$$\Rightarrow -i \sin i\theta = -i\left(i\theta - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} - \frac{(i\theta)^7}{7!} + \cdots\right)$$

$$= \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \cdots = \sinh\theta.$$

To summarize: $\exp(i\theta) = \cosh i\theta + \sinh i\theta$ (by 4/14 test)

$= \cos\theta + \sinh i\theta$ (by (d) above)

$= \cos\theta + (-i \sin i(i\theta))$ (by identity above)

$= \cos\theta - i\sin(-\theta)$

$= \cos\theta + i\sin\theta$ $\square$

Note: Many other methods are possible.

(f) Regardless of whether you succeeded or failed in part (e), assume that Euler's formula is valid. Now
(4) use it to prove *Euler's identity,* a famous equation that unifies the five most fundamental constants in mathematics: $e^{i\pi}+1=0$.

$$e^{i\pi}+1=\exp(i\pi)+1=(\cos\pi+i\sin\pi)+1=-1+i(0)+1=0 \quad \blacksquare$$

10. Suppose that you are interviewing for a job as a programmer with TI or one of the other large
(8) calculator manufacturers. The company is planning to develop a scientific calculator that will return the values of the hyperbolic functions (sinh, cosh, etc.) with unheard-of accuracy for a handheld calculator: error not to exceed 10^{-20} in absolute value. This is about a million times more accurate than our TI-83/84 units, which are, at best, accurate on hyperbolic functions only to about 10^{-14}.

Prove that you are the right man for the job by writing a thoughtful paragraph (complete sentences are required!) in which you describe how you would ensure that cosh(−2) would come out to have the specified precision. Put some detail in your answer so that the company can tell that you know what you are talking about. [Shorter answers are acceptable.]

The Maclaurin series for cosh(−2), namely

$$1+\frac{(-2)^2}{2!}+\frac{(-2)^4}{4!}+\frac{(-2)^6}{6!}+\cdots,$$

is a series of all positive terms. Although we cannot use the alternating series error bound, we can use the Lagrange form of the remainder to characterize the maximum absolute value of the tail in terms of a hyperbolic sine (or cosine) multiplied by $\left|\frac{(-2)^{2n+1}}{(2n+1)!}\right|$. [Actually, we can restrict our attention to a hyperbolic cosine term with a factor of $\frac{2^{2n+2}}{(2n+2)!}$, since all the "sinh" terms in the Maclaurin expansion of cosh x are zero anyway... but we did not learn that until after the test. For this test, it suffices to know that AST does ~~not~~ work, but something else does work, in order to precompute how many terms would be needed in order to satisfy accuracy within 10^{-20}.] For sufficiently large n, which we can readily calculate, the required accuracy will be guaranteed. The number of terms actually involved in the estimate will be n+1.