Mr. Hansen
HW due 4/12/010
(excerpt)

#26. [Note: In, #13, we saw that Maclaurin coefficients of $n!$ produced a series that converges only at $x=0$, while in #24 we saw that Maclaurin coefficients of $\frac{1}{n^n}$ produced a series that converges on $\mathbb{R}$. The question is: Who wins? Where will a Maclaurin series with coefficients of $\frac{n!}{n^n}$ converge?]

Ratio technique: $$\lim_{n\to\infty}\left|\frac{t_{n+1}}{t_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^{n+1}}\cdot x^{n+1}\cdot\frac{n^n}{n!\,x^n}\right|$$

$$= |x|\lim_{n\to\infty}\frac{n+1}{(n+1)^n(n+1)}\cdot\frac{n^n}{1}$$

$$= |x|\underbrace{\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n}_{\text{Call this "}L\text{."}}$$

Now, $$\ln L = \ln\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty}\ln\left(\frac{n}{n+1}\right)^n \qquad \text{by continuity}$$

$$= \lim_{n\to\infty} n\ln\frac{n}{n+1} \qquad \text{by prop. of logs}$$

$$= \lim_{n\to\infty} n\left(-\ln\frac{n+1}{n}\right) \qquad \text{" " " "}$$

$$= \lim_{n\to\infty} n\left(-\ln\left(1+\frac{1}{n}\right)\right) \qquad \text{by alg.}$$

$$= \lim_{n\to\infty}\frac{-\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \qquad \left(\text{in form } \frac{0}{0}\text{, so use L'Hôp.}\right)$$

$$= \lim_{n\to\infty}\frac{-\frac{1}{1+\frac{1}{n}}\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{-\frac{1}{n^2}}} = -1 \Rightarrow L = e^{-1} = \frac{1}{e}$$

Thus, from above, we have
$$|x|L = |x|\cdot\frac{1}{e} \overset{\text{want}}{<} 1 \text{ to ensure convergence}$$
$$|x| < e$$

Open interval of convergence for x is $(-e, e)$.