Honors AP Calculus / Mr. Hansen
4/15/2009

(8 pts.) Name: KEY
Battery bonus (Mr. Hansen's use only): ______

Test on Everything Except §12.8

Rules

- You may not write calculator notation anywhere unless you cross it out. For example, fnInt(X^2,X,1,2) is not allowed; write $\int_1^2 x^2 dx$ instead.
- Legibility and neatness count. Do not take the time to erase large sections. Instead, mark a single "X" through anything you wish to be ignored.
- All final answers should be *circled or boxed.* Simplification is not required unless the context of the problem clearly requires it. Decimal approximations in final answers must be correct to at least 3 places after the decimal point.
- Point values are shown in parentheses to the left of each question.
- If you need more room, write "CONT." and go to the Continuation/Bonus Page.

1. (36 pts., 6 pts. per part) Function $f(x)$ is defined on $(-1.5, 1.5)$ by the following Maclaurin series:

$$2+\tfrac{4}{3}x+\tfrac{8}{9}x^2+\cdots+\tfrac{2^{n+1}}{3^n}x^n+\cdots$$

(a) Prove that f converges for all x in $(-1.5, 1.5)$.

Ratio technique:

$$\lim_{n\to\infty}\left|\frac{t_{n+1}}{t_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+2}x^{n+1}}{3^{n+1}}\cdot\frac{3^n}{x^n\,2^{n+1}}\right|$$

$$= \lim_{n\to\infty}\tfrac{2}{3}|x|$$

$$= |x|\cdot\tfrac{2}{3} \overset{\text{want}}{<} 1$$

$\Rightarrow$ true when $|x| < \frac{3}{2}$, i.e.,

$x \in (-1.5, 1.5)$ $\square$

[Note: Only the open interval of convergence was addressed by the question.]

(b) Compute the exact value of $f(-1)$.

$$f(-1) = \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^n}(-1)^n$$, geometric series with first term (t_0) of 2 and common ratio $r = -\frac{2}{3}$

Note: $|r| < 1$

$$S_\infty = \frac{t_0}{1-r}$$

$$= \frac{2}{1-(-\frac{2}{3})}$$

$$= \frac{2}{\frac{5}{3}} = 2 \cdot \frac{3}{5} = 1.2$$

(c) It can be shown that $\lim_{x \to -1.5^+} f(x) = 1$, and therefore the domain of f could be extended slightly. If $x \in (-1.5, 1.5)$, we would define $f(x)$ by its Maclaurin series, and for the left endpoint, we would simply say $f(-1.5) = 1$.

However, the Maclaurin series cannot be used to compute $f(-1.5)$. Explain why not.

At $x = -1.5$, the series becomes

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{3^n}\left(-\frac{3}{2}\right)^n = \sum_{n=0}^{\infty} \frac{2^1}{3^0}(-1)^n$$

$$= 2 - 2 + 2 - 2 + 2 - 2 \ldots$$

Divergent by nth term test: $\lim_{n \to \infty} t_n \neq 0$

(d) Write the power series for $f'(x)$, showing at least four explicit terms and the general term. Please construct your general term to have x^n in it, not x to some other power.

$$f(x) = 2 + \frac{4}{3}x + \frac{8}{9}x^2 + \frac{16}{27}x^3 + \frac{32}{81}x^4 + \cdots \frac{2^{n+2}}{3^{n+1}}x^{n+1} + \cdots$$

$$f'(x) = \frac{4}{3} + \frac{16}{9}x + \frac{48}{27}x^2 + \underbrace{\cdots}_{\text{higher-degree terms}}$$

$$= \frac{4}{3} + \frac{16}{9}x + \frac{48}{27}x^2 + \frac{128}{81}x^3 + \cdots + \frac{(n+1)2^{n+2}}{3^{n+1}}x^n + \cdots$$

where n starts from 0

(e) Obtain an approximation of $f'(-1)$ by means of a degree 2 polynomial for $f'(x)$. (*Note:* This is easy if you did part (d). If you could not solve part (d), you will need to use a degree <u>3</u> polynomial for f as your starting point, not a degree 2 polynomial for f.) Show your work.

$$f'(-1) \approx \frac{4}{3} + \frac{16}{9}(-1) + \frac{48}{27}(-1)^2$$

$$= \frac{4}{3}$$

(f) Use your answers to parts (b) and (e) to write an equation of a line that approximates $f(x)$ when x is in a neighborhood of -1. If you could not compute the answer to (b) or (e), then simply write b or e to indicate where you would insert those values.

$$y - 1.2 \approx \frac{4}{3}(x+1)$$

or $\hat{y} = 1.2 + \frac{4}{3}(x+1)$

OR $\hat{y} = \frac{4}{3}x + \frac{38}{15}$

or $\hat{y} = \frac{4}{3}x + 2.5\overline{3}$

2. (37 pts. total) Let $g(x) = \dfrac{1-\cos x}{x}$.

(a) (6 pts.) Clearly, $g(x)$ is undefined when $x = 0$. However, $\lim_{x\to 0} g(x)$ can be computed by L'Hôpital's Rule. Please do so, showing your steps clearly.

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{1-\cos x}{x} \quad \left(\tfrac{0}{0}\ \text{L'Hôp.}\right)$$

$$= \lim_{x\to 0} \frac{0-(-\sin x)}{1}$$

$$= \frac{0}{1} = 0$$

(b) (6 pts.) Write $g(x)$ as a power series centered about $x = 0$. Raise your hand if you would like to purchase an answer or if you would like to have your answer checked, since part (b) is used extensively in the questions below.

$$\frac{1-\cos x}{x} = \frac{\cancel{1} - \left(\cancel{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}{x}$$

$$= \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \cdots$$

(c) Compute the limit in part (a) without using L'Hôpital's Rule. (Use power series instead.)
(6 pts.)

Since there is no constant term in

$\frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \cdots$, the evaluation at $x=0$

is 0 by inspection.

The series clearly converges for all $x \in \mathbb{R}$, since

$$\lim_{n\to\infty} \left|\frac{t_{n+1}}{t_n}\right| = \lim_{n\to\infty} \left|\frac{x^2}{[2(n+1)]!} \cdot (2n)!\right| = |x^2| \cdot 0 = 0 < 1$$

for all $x \in \mathbb{R}$.

Since series for $g(x)$ converges for all $x \in \mathbb{R}$, $g'(x)$ can be found by differentiating term-by-term. Thus g' exists on $\mathbb{R} \setminus \{0\} \Rightarrow$ g cont. on $\mathbb{R} \setminus \{0\} \Rightarrow \lim_{x\to 0} g(x) =$ the value above by def. of cont.

(d) Prove that your power series in part (b) satisfies all three requirements of the Alternating
(9 pts.) Series Test when $x \in [-1,1]$. (*Note:* It is possible to show that AST is eventually satisfied for any real x, but that requires more work. Today, you need only show that AST is satisfied for x between -1 and 1, inclusive.)

- Strictly alternating in sign (by inspection) [except for trivial case of $x=0$, for which series clearly converges]
  If $x<0$, signs are $-,+,-,+,$ etc.
  If $x>0$, signs are $+,-,+,-,$ etc.
- Strictly decreasing in abs. value since
  $|\text{numerator}| \le 1$ and $|\text{denominator}|$ is increasing
- $$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n)!} = 0$$ since

  $|\text{numerator}| \le 1$ and denominator increases without bound

(e) (10 pts.) Use the AST error bound to determine the number of nonzero terms needed in your power series for $g(x)$ in order to estimate $g(0.8)$ with an absolute error of less than 0.000001. Justify your answer. Note that for full credit, you must pretend that the true value of $g(0.8)$ is not known in advance (since obviously, you could otherwise answer the question simply by comparing $g(0.8)$ against various polynomials until you had one that came close enough).

After n terms, the first omitted term is

$$\frac{(-1)^{n+2}\, x^{2(n+1)-1}}{(2(n+1))!}, \text{ whose magnitude is}$$

a bound for the error.

$$\left|\frac{(-1)^{n+2}\, x^{2n+2-1}}{(2n+2)!}\right| = \frac{(.8)^{2n+1}}{(2n+2)!} \overset{\text{want}}{<} .000001$$

First true when $n = 4$.

$\therefore$ We need at least 4 terms.

3-5. **"Always/Sometimes/Never" (3 pts. each)**
Write A if the statement is always true, S if it is sometimes true, or N if it is never true. There is no partial credit.

S 3. A Taylor series is a Maclaurin series.

S 4. A real-valued power series that has an interval of convergence that is a proper subset of $\Re$ has convergence at exactly one of the two endpoints of that interval.

A 5. If x is a real number, then the series $x+\frac{x^2}{2^2}+\frac{x^3}{3^3}+\frac{x^4}{4^4}+\cdots=\sum_{n=1}^{\infty}\frac{x^n}{n^n}$ converges absolutely.

6. (12 pts.) Prove Euler's Formula: For any complex number z, $e^{iz} = \cos z + i\sin z$.

Note: For full credit and full rigor, you should know several facts that are listed below. If these confuse you, please ignore them. A less rigorous proof (i.e., with unjustified rearrangement of terms) will have only a 2-point penalty, and since there are 102 points possible on today's test, that is really no penalty at all.

- Rearrangement of terms within an infinite series is not permitted in general. However, it is permitted for *absolutely convergent* series. Thus, if you perform any rearrangement of terms, you should first justify or prove absolute convergence so that the rearrangement can be considered rigorous.
- The complex absolute value (modulus), $|z|$, is a generalization of the real-number absolute value with which you are well acquainted. All you need to know for today is this: Just as a factor of ± 1 inside the absolute value bars does not affect real-number absolute value, a factor of ± 1 or $\pm i$ inside the complex modulus bars does not affect complex modulus.
- A complex series converges absolutely iff its series of moduli converges.
- Exponentials, sines, cosines, hyperbolic sines, and hyperbolic cosines converge absolutely on the complex plane. You may treat this as a given.

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = 1 + \frac{iz}{1!} - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} + \cdots$$

(absolutely convergent on $\mathbb{C}$)

rearrangement OK since absolutely convergent on $\mathbb{C}$ by the givens

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)$$

$$= \cos z + i(\sin z)$$

$\square$

Continuation/Bonus Page

Use the space below if you ran out of room on an earlier page.

BONUS (1 point) Prove the "it can be shown" assertion in question 1(c).

For all $x \in (-1.5, 1.5)$, $f(x)$ is a geometric series with first term 2 and common ratio $\frac{2x}{3}$.

Thus $f(x) = \dfrac{2}{1-r} = \dfrac{2}{1-\frac{2x}{3}} = \dfrac{6}{3-2x}$ for $x > -1.5$ and x arbitrarily close to -1.5 from the right.

$$\therefore \lim_{x \to -1.5^+} f(x) = \lim_{x \to -1.5^+} \frac{6}{3-2x} = \frac{6}{3-(-3)} = 1$$

$\square$