Algebra II / Mr. Hansen
2/18/2016

Name: KEY (B)

Test on Chapter 5

Graphing calculator and PENCIL are required. If you need more room for rough scratch work, use the final two pages, which will not be graded. If you need more room for work that is to be graded, write OVER and continue on the reverse side. You may use the Rule of 72 once if you wish. Circle all answers.

Please print your numbers like this:

9	8	7	6	5	4	3	2	1	0

Nonconforming numbers may be misinterpreted, resulting in loss of points. For most students, it is 4 and 9 that cause trouble. European-style 1 and crossed 7 are also acceptable. A "closed-top pointy 4" is also acceptable. *Note:* All answers, unless otherwise stated, must be correct to at least 3 significant digits.

Part I: Logarithmic Detective Work (5 pts. each)

Instructions: Given that log 2 $\approx$ 0.3, estimate each of the following. No credit (**zero points**) for answers that are not supported by work. Be sure to draw a distinction between the equals sign (=) and the approximately-equals sign ($\approx$). The first one is done for you as an example. You are permitted to use the answers for #1 through #3 as "building blocks" if you wish.

1. log 20

   Solution: $\log 20 = \log(2\cdot 10) = \log 2 + \log 10 \approx 0.3 + 1 = \boxed{1.3}$

2. log 5 $= \log \frac{10}{2} = \log 10 - \log 2 \approx 1 - .3 = .7$

3. log 8 $= \log 2^3 = 3 \log 2 \approx 3(.3) = .9$

4. log 40 $= \log(4\cdot 10) = \log 4 + \log 10 = \log 2^2 + 1 = 2\log 2 + 1$

   $\approx 2(.3) + 1 = 1.6$

   ALT. METHOD: $\log 40 = \log(5\cdot 8) = \log 5 + \log 8 \approx .7 + .9 = 1.6$

Part II: Creative Work (8 pts. each)

5. Describe how to estimate 70^{90} to the nearest power of 10. The only tool you may use is your calculator. (Do use it.) Complete sentences are required for full credit. Circle your final answer.

$\log 70^{90} = 90 \log 70 \approx 166.0588$

$\therefore 70^{90} \approx 10^{166}$

ALT. METHOD 1

$70^{90} = (7\cdot 10)^{90} = 7^{90}\cdot 10^{90} \approx (1.145\cdot 10^{76})\cdot 10^{90} \approx 10^{166}$

ALT. METHOD 2

$70^{90} = (70^{45})^2 \approx (1.07\cdot 10^{83})^2 \approx 10^{166}$

6. Prove that the base-16 logarithm of any positive real number is always one-fourth as large as the base-2 logarithm of the same number. You may assume that the change-of-base formula is valid.

Given: $x > 0$

Prove: $\log_{16} x = \frac{1}{4}\log_2 x$

Proof: $\log_{16} x = \frac{\log_2 x}{\log_2 16}$ (by change-of-base)

$= \frac{\log_2 x}{4}$ (since $\log_2 16 = 4$)

$= \frac{1}{4}\log_2 x$ ∎

Part III: Criminal Forensics (20 pts.)

7. Newton's Law of Cooling: $T(t) = T_S + (T_0 - T_S)\, e^{-kt}$, where T_S = surrounding temperature, T_0 = initial temperature of object.

Fred Friendly was feeling feverish at home (in his flat on the fifth floor) on the morning of February 15. His body temperature was 100° F. when a bullet from a fraudster found Fred and felled him. A famously frugal man, Fred always kept the furnace in his house set to 55° F. As he lay dying, Fred frantically fumbled and wrote a note saying that his body temperature was 100° F., but he foolishly forgot to record the time or the name of the fraudster who killed him. When police arrived at 11:00 a.m., Fred's corpse had a temperature of 89° F., and an hour later, it had cooled to 83° F. Compute the time of death to the nearest half-hour. Show all work clearly. *Hint:* For full credit, you have to find the value of k. However, if that baffles you, check here ☐ and use the value 0.2 for k. (That's not correct, but it's close enough to give you the correct final answer.)

Model: $T(t) = T_S + (T_0 - T_S)e^{-kt}$ where $T_0 = 89$, $T_S = 55$

Note: $t = 0$ denotes arrival of police.

$T(t) = 55 + (89 - 55)e^{-kt}$

$T(t) = 55 + 34e^{-kt}$

We know $83 = 55 + 34e^{-k(1)}$

$28 = 34e^{-k}$

$\ln\frac{28}{34} = \ln e^{-k}$

$k = -\ln\left(\frac{28}{34}\right) \approx .194156\ldots$

Now, plug this value for k back into the model:

$T(t) = 55 + 34e^{-kt}$

Plug in $T(t) = 100$ and solve for t:

$100 = 55 + 34e^{-kt}$

$45 = 34e^{-kt}$

$\frac{45}{34} = e^{-kt}$

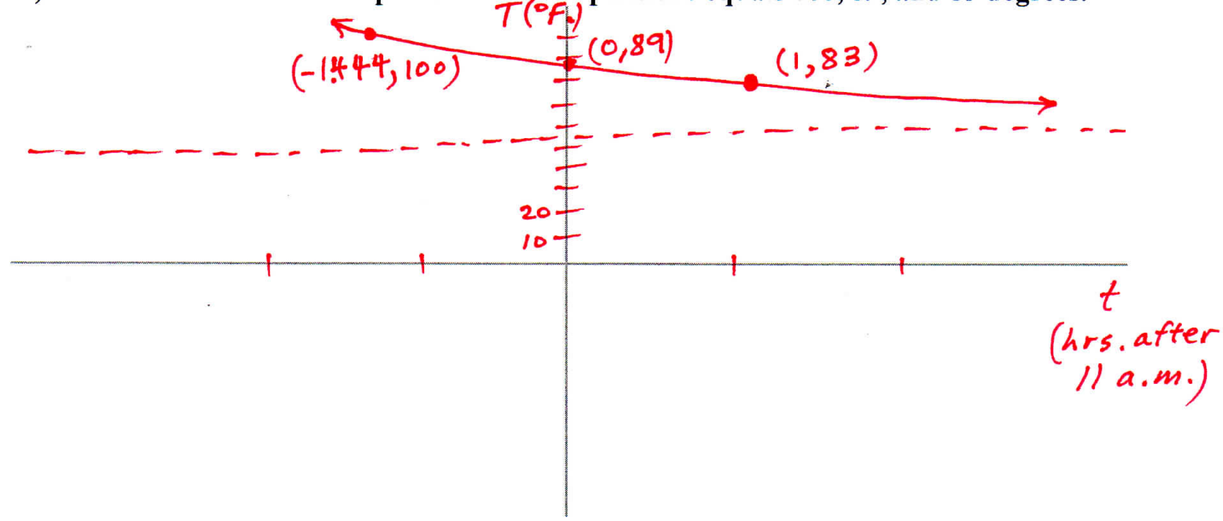
$\ln\left(\frac{45}{34}\right) = -kt$

$t = \frac{\ln\left(\frac{45}{34}\right)}{-k} \approx -1.444$

The time is 1.444 hrs. before police arrived, or about 9:30 a.m.

Part IV: Graphing (15 pts. each)

8. Sketch the graph of the $T(t)$ curve from problem #7. Be sure to label your horizontal axis as t and your vertical axis as temperature (° F.). You may use $t = 0$ to denote either the time of death or the time at which the police arrived—your choice, but be sure to indicate which choice you made. For k, use the value you found in #7. (If you were unable to compute k, use the approximate value 0.2.) **Mark coordinates for the points where temperature equals 100, 89, and 83 degrees.**



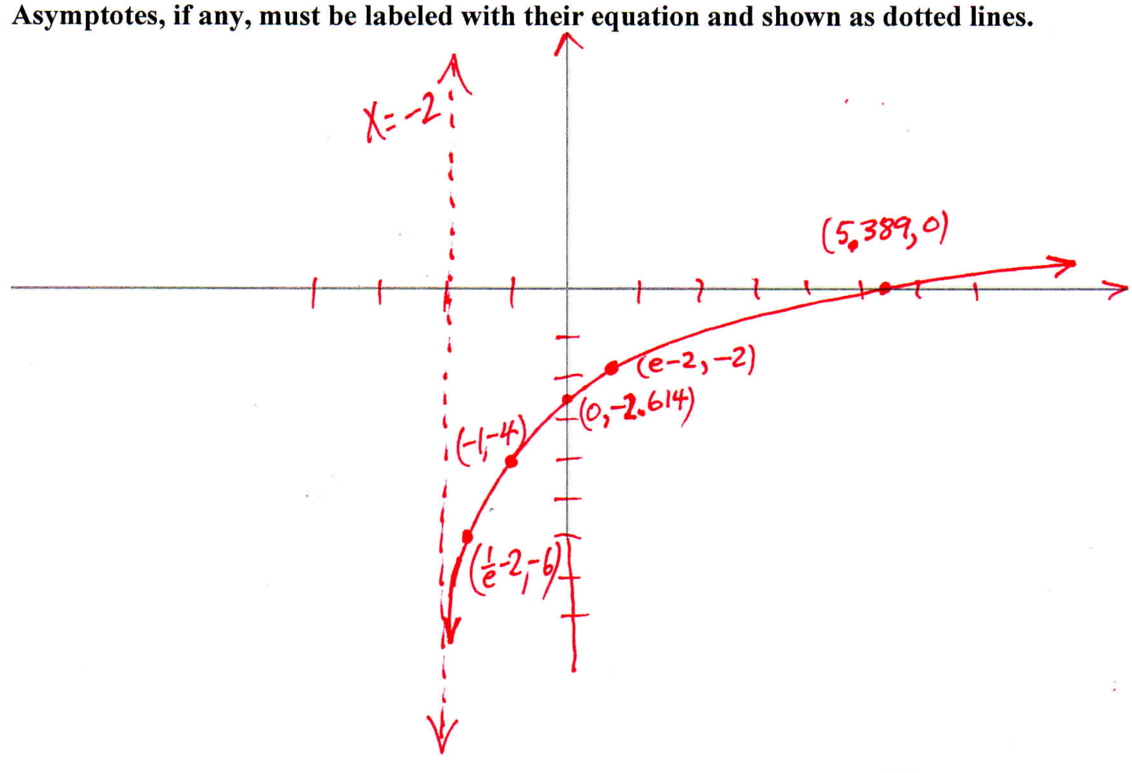
9. Explain how to sketch the curve $y = 2\ \ln\,(x + 2) - 4$. (*Note:* No credit for saying, "Plug it into my calculator and make a neat transcription onto paper." You have to describe the steps or transformations that you would perform.) Write your explanation here:

(1) Shift the standard curve for $y = \ln x$ LEFT 2 units.

(2) Expand (dilate) by vertical factor of 2: higher positives, lower negatives.

(3) Shift downward by 4 units.

Sketch the graph and mark coordinates for the points where x equals -1, $e - 2$, and $\frac{1}{e} - 2$. Asymptotes, if any, must be labeled with their equation and shown as dotted lines.



Part V: Growth/decay problems (10 pts. each)

10. A $15,000 investment has a yield of approximately 6%, compounded annually. Compute the time required for the investment to be worth $30,000. Explain your reasoning adequately.

STANDARD METHOD

$A = A_0\left(1+\frac{r}{n}\right)^{nt}$

$30{,}000 = 15{,}000\left(1+\frac{.06}{1}\right)^{1t}$

$\frac{30{,}000}{15{,}000} = 1.06^t$

$2 = 1.06^t$

$\log_{1.06} 2 = t = \frac{\ln 2}{\ln 1.06}$

$t \approx 11.896$ years

ALT. METHOD

Use Rule of 72 to get $\frac{72}{6} \doteq 12$ years

[Note: cannot use Pe^{rt} model]

11. A 10 kg lump of radioactive thessoramide (a fictitious substance) has a decay rate of 9% per year. Compute the half-life, i.e., the time needed for the 10 kg lump to decay to 5 kg. Explain your reasoning adequately.

STANDARD METHOD

$y = Pe^{rt}$

$5 = 10e^{-.09t}$

$\frac{5}{10} = e^{-.09t}$

$\ln\left(\frac{5}{10}\right) = -.09t$

$t = \frac{\ln\left(\frac{1}{2}\right)}{-.09}$

$t \approx 7.702$ years

ALT. METHOD 1*

$y = 0.91^t(10)$

$5 = 0.91^t(10)$

$\frac{5}{10} = 0.91^t$

$\ln\left(\frac{5}{10}\right) = t \ln 0.91$

$t = \frac{\ln\left(\frac{5}{10}\right)}{\ln 0.91}$

$t \approx 7.350$ years

ALT. METHOD 2

Use Rule of 72 to get $\frac{72}{9} = 8$ years

*Assumes annual compounding, not continuous compounding, of the decay rate. Radioactive decay follows continuous decay, which means that the Pe^{rt} model is better.