Honors AP Calculus / Mr. Hansen
12/14/2011

Name: KEY
Mr. Hansen's use only: ________

7

Test (100 pts.): Calculator Permitted

Part I: Fill-Ins.
Write the word or phrase that best fits each blank. For names, a last name is sufficient.

4 1. The term *chaos* may be mathematically defined as extreme sensitivity to initial conditions

4 2. A mathematician who died in 2010 is remembered as the man most closely associated with the science of fractals. His name was Benoit Mandelbrot.

8(2ea.) 3. IVT, EVT, and FTC1/2 all have as their hypothesis that we have a continuous function on a(n) closed interval. MVT has the additional hypothesis that the function is differentiable on a(n) open interval.

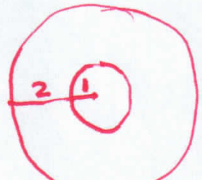
4 4. "A point of continuity at which f'' changes sign" is a suitable definition of a point of inflection.

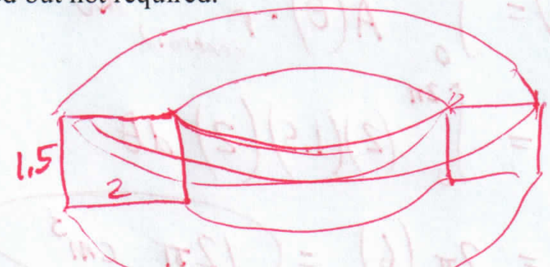
Part II. Volumes.

5. The green Play-Doh object is a ring having rectangular cross sections. You can think of it as "a hockey puck with a hole in it." The rectangular cross sections are 2 cm wide by 1.5 cm high. The inner radius of the object is 1 cm, and the outer radius is 3 cm. Compute the volume by (a) plane slicing, (b) cylindrical shells, and (c) radial slicing. You may leave π in your answers, or you may give approximate answers to at least 3 decimal places. Most of the points are in the setups of the integrals, so be sure to show adequate work: unplugged formulas, plugged formulas, and the work (if any) leading to a final answer, which must be circled or boxed in each case. Sketches are recommended but not required.

4 setup
3 answer
1 units (cm^3)

(a) Plane Slicing





$$A(y) = \pi R^2 - \pi r^2 = \pi(3^2 - 1^2) = 8\pi$$

$$V = \int_0^{1.5} A(y)\,dy = 1.5(8\pi) = 12\pi \text{ cm}^3$$

(b) Cylindrical Shells

$4 \} $
$3 \} $ as in (a)
$1 \} $

$$V = \int_1^3 2\pi r h\, dr = 2\pi \int_1^3 r(1.5)\, dr$$

$$= 2\pi \cdot \frac{1.5 r^2}{2}\Big|_1^3 = 2\pi\left(\frac{1.5(9)}{2} - \frac{1.5(1)}{2}\right)$$

$$= 2\pi\left(\frac{13.5 - 1.5}{2}\right) = 2\pi(6) = 12\pi\ cm^3$$

(c) Radial Slicing

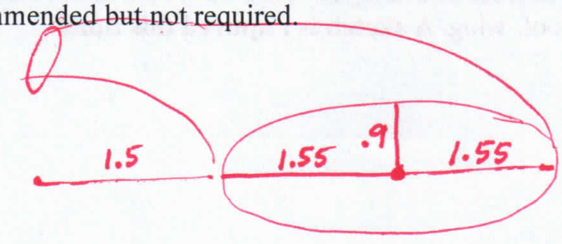
4 setup
1 ID of $r_{centroid}$
3 answer
1 units (cm^3)

$$V = \int_0^{2\pi} A(\theta) \cdot r_{centroid}\, d\theta$$

$$= \int_0^{2\pi} (2)(1.5)(2)\, d\theta$$

$$= 2\pi(6) = 12\pi\ cm^3$$

$5\frac{1}{2}$. On Tuesday, Dec. 13, Mr. Hansen attended a Christmas musical party at which a blueberry coffee cake was served. Mr. Hansen was given exactly one quarter of the cake to take home (90 degrees of pure deliciousness). The cross sections are, let us say, ellipses having major axis of 3.1 inches and minor axis of 1.8 inches. The inner radius of the coffee cake, before it was cut, was 1.5 inches. Compute the volume of the *one-quarter cake* that is available for inspection. A sketch is recommended but not required.



4 integrand
2 limits
2 answer
1 units(in^3)

$$V = \int_0^{\frac{\pi}{2}} A(\theta) \cdot r_{centroid} \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \pi r_1 r_2 \, (3.05) \, d\theta$$

$$= \frac{\pi}{2} \cdot \pi \cdot 1.55 \cdot 0.9 \cdot 3.05$$

$$= 2.127375\,\pi^2 \approx 20.996 \text{ cu. in.}$$

6. In #~~4~~5, suppose that a factory is experimenting with various types of "hockey pucks with holes in them." Let r denote the inner radius, R the outer radius, and h the height (thickness) of the puck. Prove by any calculus-based method you wish that the volume equals $\pi h(R^2 - r^2)$ cubic units. *Note:* This formula is easy to prove with freshman geometry, and you may certainly use that to cross-check if you wish, but you will not earn any points unless you use a calculus-based proof, wlog. **A sketch is required this time.**

3 sketch
3 integrand
2 limits
$\frac{1}{2}$ "QED" or $\square$

View from above:

$$A = \pi R^2 - \pi r^2$$

Plane slicing:

$$V = \int_0^h A(y)\,dy$$

$$= \int_0^h \underbrace{(\pi R^2 - \pi r^2)}_{\text{constant}}\,dy$$

$$= \pi h(R^2 - r^2) \qquad \square$$

7. In #6, let us imagine that the factory *insists* that R, the outer radius, be fixed at 5 cm. However, the inner radius r and height h are allowed to vary in such a way that the volume is a fixed constant value of 70 cm^3. In order to minimize the cost of the paint that must be applied to every exposed surface, the factory wishes to minimize the total surface area of the manufactured "hockey pucks with holes in them." What do we call surface area in this situation? Answer: the objective function.

(a) 2

(b) Let f denote the surface area function. Write f as a function of r alone, and show the work leading to your answer. Simplification is not required.

Given: $V = \pi h(R^2 - r^2) = \pi h(25 - r^2) = 70$

$$h = \frac{70}{\pi(25-r^2)}$$

4 uses constraint to solve for h

4 $f(r)$ expression

1 circle

$$\therefore f(r) = \underbrace{\text{top} + \text{bottom}} + (\text{outer circumf.})\cdot h + (\text{inner circumf.})\cdot h$$

$$= 2\pi(R^2 - r^2) + 2\pi R h + 2\pi r h$$

$$= 2\pi(25 - r^2) + 2\pi(5)\cdot\frac{70}{\pi(25-r^2)} + \frac{2\pi r\cdot 70}{\pi(25-r^2)}$$

(c) As you surely noticed, $f(r)$ is a rather messy function. Without actually minimizing f, write a **grammatically correct paragraph** in which you describe how you would go about minimizing f. Standard abbreviations are permitted. Be sure to state the domain for r, and be sure to describe how the r value you expect to find actually gives a minimum value for f. The "hole in the hockey puck" is not allowed to vanish completely, nor can the hole extend all the way to the outer radius. **Do not actually do the minimization. That would take too long.**

10 if essentially OK

(−2 for ea. serious error, such as not stating $\mathcal{D}_r$ or not explaining how to find crit. pts.)

$\mathcal{D}_r = (0, 5)$

Compute $f'(r)$ from expression above.

Since $f'(r)$ exists on $\mathcal{D}_r$, we need only find the root(s) of $f'(r)$ on $\mathcal{D}_r$. These are all the critical pts.

[There are 2 roots, one at $r_1 \approx 0.56689$, and one at $r_2 \approx 3.106135$, but these were not requested.]

2 domain $(0,5)$

2 find crit. pts. (plural) by setting $f'(r)=0$

2 use 1st or 2nd deriv. test, sign analysis

2 mention endpts.

2 compare vals.

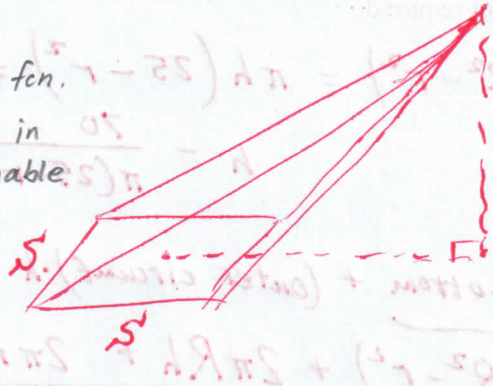
For each root, compute $f(r_i)$, and use sign analysis to verify that f' changes sign from − to + there. (If not, r_i does not give a min.) No end points to check.

If there is only one such value for r_i, then it gives the global min.; otherwise, compare all such $f(r_i)$ and choose the least.

★ Important: Also confirm that f is not lower as $r \to 0$ or as $r \to 5$.

8. In geometry class, you learned that the volume of a right regular pyramid with square base equals $\dfrac{s^2h}{3}$, where s = side length of the base and h = altitude = perpendicular distance from apex to the plane containing the base. Prove, using the calculus, that the formula is equally valid for any "oblique square pyramid" of the type sketched on the board. *Hint:* The side length of the square cross sections tapers linearly from s at the base to 0 at the apex. A sketch is optional.

4 side length fcn.
4 integrand in any reasonable form
2 limits
2 work
$\frac{1}{2}$ "QED" or □



Linear function:

height (z)	side length
0	s
h	0

slope $= \dfrac{\Delta \text{side}}{\Delta \text{height}} = -\dfrac{s}{h}$

side length $= -\dfrac{s}{h}(\text{height} - h)$

Let the square base lie in the xy-plane, and let the z-axis be vertical, perpendicular to the base.

$$V_{pyramid} = \int_0^h A(z)\,dz = \int_0^h (\text{side length})^2\,dz$$

$$= \int_0^h \left(-\frac{s}{h}(z-h)\right)^2 dz$$

$$= \frac{s^2}{h^2}\int_0^h \left(z^2 - 2zh + h^2\right) dz$$

$$= \frac{s^2}{h^2}\left(\frac{z^3}{3} - z^2h + h^2z\right)\Bigg|_0^h$$

$$= \frac{s^2}{h^2}\left(\frac{h^3}{3} - h^3 + h^3\right)$$

$$= \frac{s^2h}{3} \quad \square$$