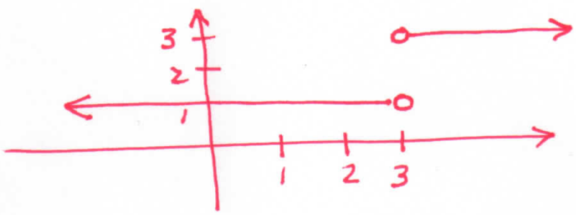
AP Calculus AB / Mr. Hansen
10/12/2005

Name: KEY

Sample Test

Time limit: 32 minutes (48 minutes for extended time).
Calculator is permitted for all problems.
Do not spend too much time on any single problem.
Clues to some problems that you cannot solve initially may be found elsewhere in the test.
All problems are free-response.
Use appropriate mathematical notation. Legibility counts.

1. Sketch a function *f* that is not defined at $x = 3$ and has a step discontinuity there.



2. Explain clearly why, even if #1 is modified so that the function *is* defined at $x = 3$, the function cannot possibly be differentiable there. Try to consider at least two different cases, i.e., different ways in which the function can be defined at $x = 3$. A formal proof is not expected. At least one of the L.H. or R.H. deriv. will always be infinite.

NOTE: Other cases are possible.

Case I: Let $f(3) = 1$. Then the L.H. deriv. at $x = 3$ is 0, but the R.H. deriv. is DNE $(+\infty)$ since $\lim_{h\to 0^+} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^+} \frac{3-1}{h} = +\infty$.

Case II: Let $f(3) = 2$. Then the L.H. deriv. at $x = 3$ is $\lim_{h\to 0^+} \frac{f(3)-f(3-h)}{h} = \lim_{h\to 0^+} \frac{2-1}{h} = +\infty$, and the R.H. deriv. is also DNE $(+\infty)$ as in Case I, except with $3-2$ in the numerator.

3. State the IVT. Standard abbreviations are encouraged.

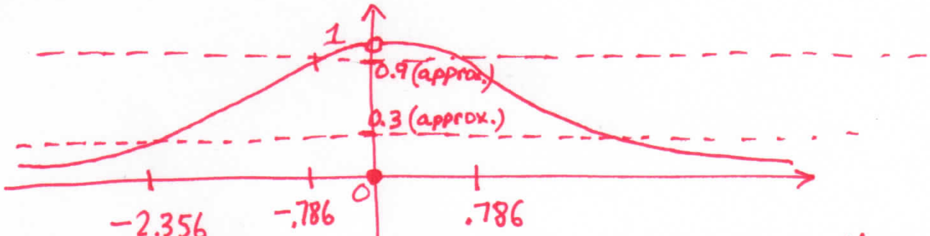
f cont. on $[a, b] \Rightarrow \Big(\forall y$ strictly between $f(a)$ and $f(b)$, $\exists x \in (a, b) \ni f(x) = y\Big)$

4. Let $y = f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Explain clearly in what way(s) the given function violates the hypotheses of the IVT for any closed interval that includes 0.

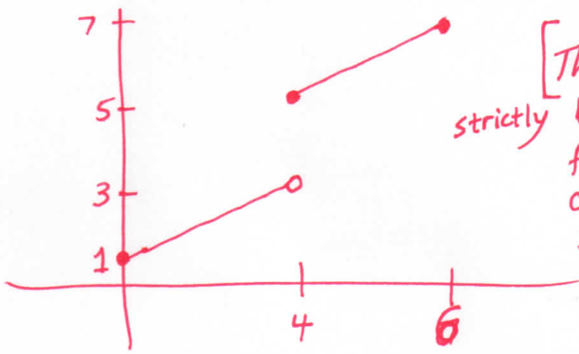
Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (proved in textbook), and since $1 \neq f(0) = 0$, f is not continuous at $x = 0$.
IVT's hypotheses are: function continuous on a closed interval. Thus any interval containing 0 will not satisfy these.

5. Prove, rigorously, that even though function f in #4 violates the hypotheses of the IVT for $[-2.356, 0.786]$, a closed interval that includes 0, the IVT conclusion is nevertheless true. You may find a sketch to be helpful, but a sketch is not required.



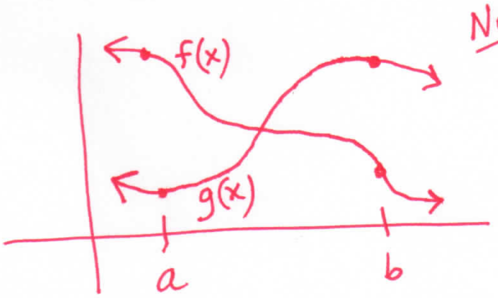
Since f is the quotient of odd functions (except at the origin), f is even. Therefore, $f(.786) = f(-.786)$, and all y-values between $f(-2.356) \approx .9$ and $f(.786) \approx .3$ are actually attained on a smaller interval, namely $[-2.356, -.786]$, on which f is continuous. Therefore, let $a = -2.356$ and $b = -.786$, and IVT applies: any y-value strictly between $f(-2.356)$ and $f(-.786)$ has at least one x-value in $(-2.356, -.786)$, and hence in $(-2.356, 0.786)$, such that $y = f(x)$. $\square$

6. Sketch a function that violates both the hypothesis and the conclusion of the IVT.



[There are y-values strictly between 1 and 7 for which no corresponding x-value exists on $(0, 6)$.]

7. Sketch two functions f and g that are continuous on $\Re$ and have the property that $f(a) > g(a)$, while $f(b) < g(b)$, where $a < b$. Let $h(x) = f(x) - g(x)$, and take it as a given that the difference of continuous functions is continuous. Prove that $\exists x \in (a, b) \ni f(x) = g(x)$.



Note: $h(a) > 0$; wlog let $Q = h(a) > 0$.

Also, $h(b) = f(b) - g(b) < 0$; wlog define $R = h(b) < 0$.

Since f and g are cont., h is also (given).

Thus by IVT, $\forall y \in (R, Q)\ \exists x \in (a, b) \ni y = h(x)$.

However, $R < 0 < Q$.

Therefore, we may take $y = 0$. Thus $\exists x \in (a, b) \ni y = 0 = h(x)$. But if $0 = h(x)$, then by def. of h we have $f(x) = g(x)$.

$\square$

8. Let $h(x) = x^3 - 10x^2 + 3x + 31$.

(a) Suppose that someone doubts that this polynomial function is continuous. (We already know that all polynomials are continuous, but assume that there is someone who did not get the memo.) Use the definition of continuity (all 3 parts) to prove that h is continuous on [2, 3]. Limits can be proved using limit properties; no need for epsilons and deltas here.

Let $a \in [2,3]$ be specified. Since a is wlog, we need only check the def. for a.

① Does $h(a)$ exist in $\mathbb{R}$? Clearly $h(a) = a^3 - 10a^2 + 3a + 31 \in \mathbb{R}$ by field closure properties.

② Does $\lim_{x\to a} h(x)$ exist in $\mathbb{R}$? Yes, since $\lim_{x\to a} h(x) = \lim_{x\to a}(x^3 - 10x^2 + 3x + 31)$

$= \lim x^3 - \lim 10x^2 + \lim 3x + \lim 31$ [limit of sum/diff.]

$= (\lim x)(\lim x)(\lim x) - 10(\lim x)(\lim x) + 3\lim x + \lim 31$ [limit of prod., lim of const. mult.]

$= a \cdot a \cdot a - 10a \cdot a + 3a + 31$ [lim. of ident. fcn.]

$= a^3 - 10a^2 + 3a + 31 \in \mathbb{R}$ [by field closure properties]

③ Does answer to #1 equal answer to #2? Yes, by computation. $\square$

(b) Compute the sign (+ or –) of $h(2)$ and of $h(3)$. (Exact values of $h(2)$ and $h(3)$ are not required.)

$h(2) = 5 > 0$ (positive)

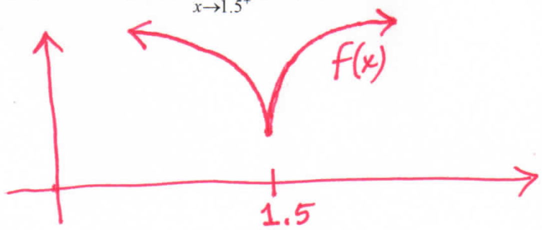
$h(3) = -23 < 0$ (negative)

(c) What does IVT allow you to conclude from (a) and (b) regarding roots?

Since 0 is strictly between $h(2)$ and $h(3)$, IVT says $\exists x \in (2,3) \ni 0 = h(x)$.

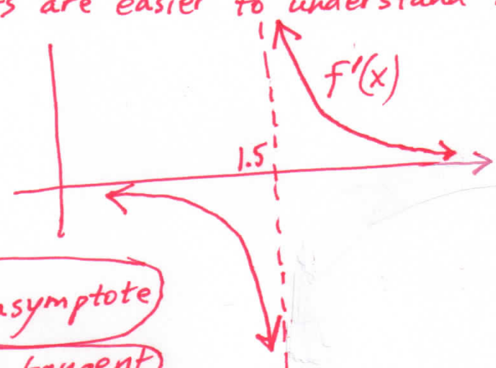
In other words, h has a root strictly between $x=2$ and $x=3$.

9.(a) Sketch a continuous function f that has a cusp at $x = 1.5$, such that f is differentiable on $\Re \setminus \{1.5\}$, $\lim_{x\to 1.5^-} f'(x) = -\infty$, and $\lim_{x\to 1.5^+} f'(x) = +\infty$.



(b, c) What role does the line $x = 1.5$ play for f' ? for f itself?

These answers are easier to understand if we sketch f':



(b) vertical asymptote

(c) vertical tangent

10. Find $\frac{dy}{dx}$ if $y = 3\sqrt[3]{3x^3 - 3\cos^3(3\sin^3 3x^3 - 3x^3 + \frac{3}{x})^3}$.

$$y' = 3 \cdot \tfrac{1}{3}\left(3x^3 - 3\cos^3\left(3\sin^3 3x^3 - 3x^3 + \tfrac{3}{x}\right)^3\right)^{-\frac{2}{3}}$$

$$\cdot \Big[9x^2 - 9\cos^2\left(3\sin^3 3x^3 - 3x^3 + \tfrac{3}{x}\right)^3 \cdot 3\left(3\sin^3 3x^3 - 3x^3 + \tfrac{3}{x}\right)^2 \left(-\sin\left(3\sin^3 3x^3 - 3x^3 + \tfrac{3}{x}\right)^3\right)$$

$$\left(9\sin^2 3x^3 \cdot \cos 3x^3 \cdot 9x^2 - 9x^2 - 3x^{-2}\right)\Big]$$

(verified correct with WolframAlpha.com)

[What a mess! This is more difficult than what you would be expected to do on a real test.]

11. Solve the following differential equation for y subject to the initial condition $(2, -1)$:

$y' = \sin 3x + 2x^3 - (x-1)^4$

The term "initial condition" refers to the fact that the unknown curve for y must pass through $(2, -1)$.

$$y = \int \left(\sin 3x + 2x^3 - (x-1)^4\right) dx = \frac{-\cos 3x}{3} + 2\left(\frac{x^4}{4}\right) - \frac{(x-1)^5}{5} + C$$

When $x = 2$, $y = -1$.

$$\therefore \quad -1 = \frac{-\cos 6}{3} + 2\left(\frac{16}{4}\right) - \frac{1}{5} + C$$

$$-1 = -\frac{\cos 6}{3} + 7.8 + C$$

$$\frac{\cos 6}{3} - 8.8 = C$$

$$y = \frac{-\cos 3x}{3} + \frac{x^4}{2} - \frac{(x-1)^5}{5} + \frac{\cos 6}{3} - 8.8$$

12. In #11, compute y when $x = 2.3$. (Note that this question can be answered even if you were unable to answer #11 itself.)

$y(2.3) = 4.4976\ldots$ by plugging into #11, or

$$y(2.3) = -1 + \int_2^{2.3} \left(\sin 3x + 2x^3 - (x-1)^4\right) dx \approx -1 + 5.4976$$

$$\approx 4.498$$