Honors AP Calculus / Mr. Hansen
1/31/2011

Name: KEY
Bonus (for Mr. Hansen's use only): ________

Test through Chapter 8 (Calculator Required)

Rules

- You may not write calculator notation anywhere unless you cross it out. For example, fnInt(X^2,X,1,2) is not allowed; write $\int_1^2 x^2dx$ instead.
- Adequate justification is required for free-response questions.
- All final answers in free-response portions should be *circled or boxed.*
- Decimal approximations must be correct to at least 3 places after the decimal point.

1. At Bisneyland, there are already 200 people (mostly employees) within the park gates at 9:00 a.m. when the gates open. The function $F(t) = 230 + 640 \sin(0.9t + 0.8)$ is a good model of the net flow rate into the park, in people per hour, where t denotes elapsed time (in hours) after 9:00 a.m.

(2) (a) Find a time t for which $F(t)$ is negative.

$t = 4$, for example

(2) (b) State the clock hour (and include a.m. or p.m., please) corresponding to your answer in part (a).

$t = 4 \Leftrightarrow$ time of 1:00 p.m.

(c) Interpret what $F(t) < 0$ means, in the context of this problem, for the value of t you gave in part (a). Write a complete, grammatically correct sentence.

(1+2+1+2 = 6)

(1) $F(4) = -379.0253...$ means that at 1:00 p.m., the (2) instantan~~eous~~ net(1) flow rate of people leaving(2) the park (excess of those leaving over those entering) is about 379 people per hour, or more than 6 per minute.

(d) Let $P(t)$ denote the population of people within the park gates at time t, where again t is measured in hours and $t = 0$ corresponds to 9:00 a.m. Find the first point of inflection for $P(t)$ on the interval [0, 7]. Justify your answer.

(4)

$P(t)$ has pts. of inflection where $F'(t) = P''(t)$ changes sign.

$F'(t) = 640 \cos(.9t + .8) \cdot (0.9)$, a sinusoid with center line on the x-axis. Thus we need only find the first root of $F'(t)$ on $[0, 7]$, which occurs when $t = .856...$, since $F'(t)$ changes from pos. to neg. there.

required later

From the given fact that $P(0) = 200$, we know $P(t) = 200 + \int_0^t F(z)dz$, so that $P(.856...) \approx 892.417$.

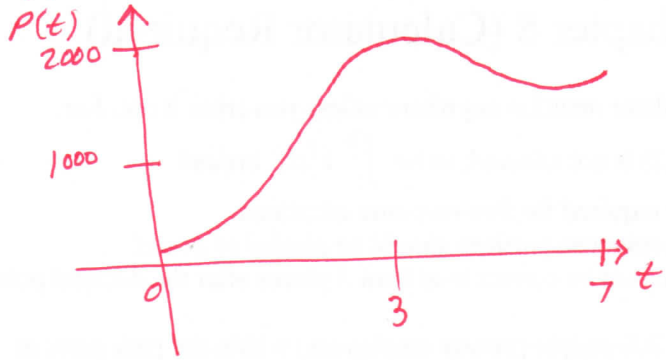
(3 + 3)

Answer: $\boxed{(0.856, 892.417)}$

[Some rubrics would accept rounding the value of $P(t)$ since $P(t)$ denotes # of people.]

(e) Find the times between 9:00 a.m. and 4:00 p.m., inclusive, to the nearest clock hour and minute, at which the population P of people in the park is maximized and minimized. Justify your answers. Please use reverse side of paper.

[$P(t)$ looks like this:



However, reading maxima off a graph is not permitted on the AP exam. Even the minimum at (0, 200), which seems obvious, must be justified.]

Instead, we must find critical points for $P(t)$, and check all such critical points as well as the endpts. ($t = 0$ and $t = 7$).

Root finder gives critical pts. (i.e., zeros of $F(t)$) at approx. $t = 3.0102119$ and $t = 5.6839858$.

There are only 4 points to check:

2 crit. pts. + 2 endpts. (2 each, 8 total)

$P(t)$ value for each (2 each, 8 total)

CONCLUSIONS in hh:mm format (2 each, 4 total)

t	$P(t)$	min.?	max.?
0	200	✓	
3.0102119	2051.388698		✓
5.6839858	1339.148546		
7	1818.647142		

Time $t = 3.0102119$ is slightly after noon, namely .0102119 hours (or about .6127 minutes) after noon.

MIN. value of $P(t)$ is at 9:00 a.m.
MAX. value of $P(t)$ is at about 12:01 p.m.

Key 2

2. A *lozenge,* believe it or not, is defined to be a special type of rhombus. Let L denote the lozenge in the xy-plane defined by the four vertices $(0, 0)$, $(1, 1+2\sqrt{2})$, $(2, 0)$, and $(1, -1-2\sqrt{2})$.

(a) Use plane slicing to compute the area of L. Be sure to indicate whether your slices are dx or dy slices. Since the area is so easy to check with geometry ($A_{\text{rhombus}} = 0.5$ times the product of the diagonals' lengths), a wrong answer is not really acceptable here. Use plane slicing, though. A sketch is required for full credit.

2 integrals or appeal to symmetry (2)

integrand(s) (4)

limits (2)

answer (2)

Using dx slices (see equations for lines below):

$$\int_0^1 \left[(1+2\sqrt{2})x - (-(1+2\sqrt{2})x)\right] dx + \int_1^2 \left[(-1-2\sqrt{2})(x-2) - (1+2\sqrt{2})(x-2)\right] dx$$

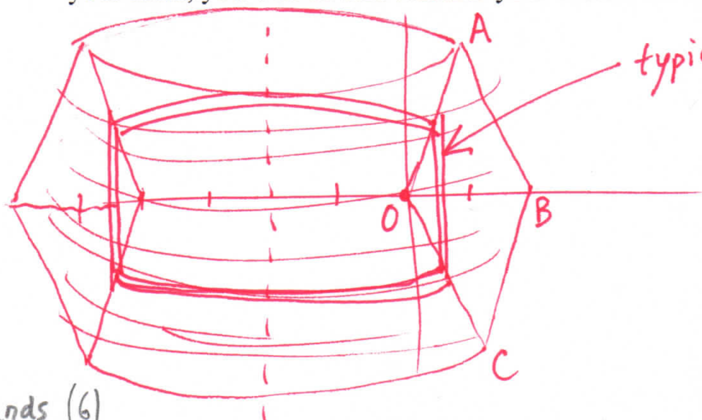
$$= 3.828\ldots + 3.828\ldots \approx 7.657 \text{ sq. units}$$

Using dy slices (see equations on next page, in part (d)):

$$\int_{-1-2\sqrt{2}}^{0} \left[2 + \frac{y}{1+2\sqrt{2}} - \frac{y}{-1-2\sqrt{2}}\right] dy + \int_0^{1+2\sqrt{2}} \left[2 + \frac{y}{-1-2\sqrt{2}} - \frac{y}{1+2\sqrt{2}}\right] dy$$

$$= 3.828\ldots + 3.828\ldots \approx 7.657 \text{ sq. units}$$

(b) Use cylindrical shells to compute the volume created when L is revolved about the line $x = -2$. A sketch is optional but highly recommended. In fact, if you show a sketch and raise your hand, you will be told whether your sketch is correct or not.



typical cylindrical shell

integrands (6)

limits (2)

answer (2)

$\overleftrightarrow{OA}$: $y - 0 = m(x-0)$

$y = (1+2\sqrt{2})x$

$\overleftrightarrow{OC}$: $y = -(1+2\sqrt{2})x$

$\overleftrightarrow{AB}$: $y - 0 = -(1+2\sqrt{2})(x-2)$

$y = (-1-2\sqrt{2})(x-2)$

$\overleftrightarrow{BC}$: $y = (1+2\sqrt{2})(x-2)$

$$V = \int_0^1 2\pi r h\, dx + \int_1^2 2\pi r h\, dx$$

$$= \int_0^1 2\pi(x+2)\left[(1+2\sqrt{2})x - (-(1+2\sqrt{x}))x\right] dx + \int_1^2 2\pi(x+2)\left[(-1-2\sqrt{2})(x-2) - (1+2\sqrt{2})(x-2)\right] dx$$

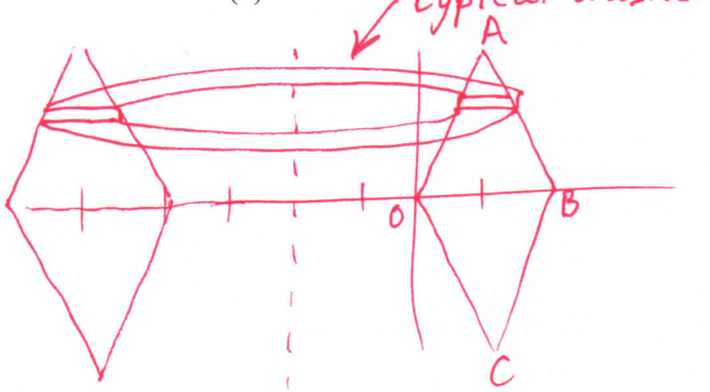
$$\approx 64.1459\ldots + 80.18239\ldots \approx 144.328 \text{ cu. units}$$

Key 3

(c) Explain why, in part (b), the integral with limits ~~2~~ 0 to ~~3~~ 1 cannot simply be doubled in order to obtain the final answer. A single sentence of explanation will suffice.

(3) The shells with larger radii (radii 3 to 4 for $x \in [1,2]$) have significantly larger volumes than those with radii 2 to 3 (i.e., $x \in [0,1]$).

(d) Use plane slicing (planes perpendicular to the y-axis) to cross-check your answer to part (b).



$\overleftrightarrow{OA}$: $y = (1+2\sqrt{2})x$

$$x = \frac{y}{1+2\sqrt{2}}$$

$\overleftrightarrow{OC}$: $y = (-1-2\sqrt{2})x$

$$x = \frac{y}{-1-2\sqrt{2}}$$

$\overleftrightarrow{AB}$: $y = (-1-2\sqrt{2})(x-2)$

$$x - 2 = \frac{y}{-1-2\sqrt{2}}$$

$$x = 2 + \frac{y}{-1-2\sqrt{2}}$$

$\overleftrightarrow{BC}$: $y = (1+2\sqrt{2})(x-2)$

$$x - 2 = \frac{y}{1+2\sqrt{2}}$$

$$x = 2 + \frac{y}{1+2\sqrt{2}}$$

<u>Upper half (doubled by symmetry):</u>

Outer radius:

$$R = 2 + \frac{y}{-1-2\sqrt{2}} - (-2)$$

Inner radius

$$r = \frac{y}{1+2\sqrt{2}} - (-2)$$

integrand (5) (6)
limits (2)
answer (2)

$$V = 2\int_0^{1+2\sqrt{2}} (\pi R^2 - \pi r^2)\, dy$$

$$= 2\pi \int_0^{1+2\sqrt{2}} \left[\left(4 + \frac{y}{-1-2\sqrt{2}}\right)^2 - \left(2 + \frac{y}{1+2\sqrt{2}}\right)^2 \right] dy$$

$\approx$ 144.328 cu. units

Key 4

(e) Use radial slicing to triple-check your answer to part (b). If you are running low on time, be sure to at least show the setup.

integrand (4)
limits (2)
answer (2)

$$V = \int_0^{2\pi} (\text{area of lozenge}) \cdot r \, d\theta$$ where $r =$ radius to centroid

$$= \int_0^{2\pi} \tfrac{1}{2} d_1 d_2 \cdot 3 \, d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}(2)(2+4\sqrt{2}) \cdot 3 \, d\theta$$

$$\approx 144.328 \text{ cu. units}$$

3.(a) What sort of figure is the polar graph $r = 2\cos\theta$ for $\theta \in [0, 2\pi]$?

(2) a circle [traced twice]

(b, c) Compute the area of the figure in part (a) using simple geometry and also using the polar area formula. Explain any discrepancy you encounter.

(2) (b) $A_{circle} = \pi r^2 = \pi \cdot 1^2 = \pi$ square units

integrand (4) limits or explan. (2) answer (1)

(c) $A_{polar} = \int_0^{\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi} \frac{1}{2}(2\cos\theta)^2 \, d\theta = \pi$ sq. units by calc.

[No discrepancy, since circle is traversed only once for limits of 0 to π.]

4. Compute the arc length (in Quadrant I *only*) of the parabola $y = 9 - x^2$. You have enough room here, but if you need more room, write OVER.

integrand (4)
limits (2)
answer (2)

Let $y = f(x)$. $L = \int_0^3 \sqrt{1 + [f'(x)]^2} \, dx = \int_0^3 \sqrt{1 + (-2x)^2} \, dx \approx 9.747$ units